OCR
Oxford Cambridge and RSA

Monday 15 May 2023

AS Level Further Mathematics B (MEI)

Y410/01 Core Pure

Worked Solutions

Printed Answer Booklet

Time allowed: 1 hour 15 minutes

You must have:
- Question Paper Y410/01 (inside this document)
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

R red level
- longer questions (6+ marks)
- higher level problem solving
- harder A Level content

A amber level
- Shorter questions (3-6 marks)
- low level problem solving
- harder AS/easier A Level content.

G green level
- Short questions (1-3 marks)
- minimal problem solving
- AS/easier A Level content.

E explanation

Q1: Matrices
Q2: Algebra
Q3: Algebra, complex numbers
Q4: Series
Q5: Complex Numbers
Q6: Matrices, Proof
Q7: Complex Numbers
Q8: Matrices
Q9: Matrices
Q10: Vectors

Grade Boundaries

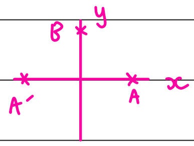
Grade	A	B	C	D	E	U
Mark / 60	44	39	34	29	25	0

G1

1 The transformation R of the plane is reflection in the line $x = 0$.

↳ y-axis

(a) Write down the matrix **M** associated with R. **[1]**



so $M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

G1

(b) Find $\mathbf{M}^2$. **[1]**

$$M^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (= I)$$

G1

(c) Interpret the result of part **(b)** in terms of the transformation R. **[1]**

Since M^2 is the identity matrix, two reflections is the equivalent of the identity transformation.

A

2 In this question you must show detailed reasoning.

The equation $x^2 - kx + 2k = 0$, where k is a non-zero constant, has roots α and β.

Find $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ in terms of k, simplifying your answer. **[4]**

First we simplify $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ using $\sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta$

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{\sum \alpha^2}{\sum \alpha\beta} = \frac{(\sum \alpha)^2 - 2\sum \alpha\beta}{\sum \alpha\beta}$$

Now we find $\sum \alpha$ and $\sum \alpha\beta$

$$\sum \alpha = -\frac{b}{a} = -\frac{(-k)}{1} = k$$

$$\sum \alpha\beta = \frac{c}{a} = \frac{2k}{1} = 2k$$

And sub these values into our expression.

So $$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{(k)^2 - 2(2k)}{2k} = \frac{k^2 - 4k}{2k}$$

$$= \frac{1}{2}k - 2$$

A

3 In this question you must show detailed reasoning.

The function $f(z)$ is given by $f(z) = 2z^3 - 7z^2 + 16z - 15$.

By first evaluating $f\left(\frac{3}{2}\right)$, find the roots of $f(z) = 0$. **[6]**

First find $f(x)$ and use factor theorem

$f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - 7\left(\frac{3}{2}\right)^2 + 16\left(\frac{3}{2}\right) - 15$

$= \frac{27}{4} - \frac{63}{4} + 24 - 15$

$= 0$ $\therefore$ by factor theorem, as $f\left(\frac{3}{2}\right) = 0$,

$\left(z - \frac{3}{2}\right)$ is a factor of $f(z)$.

$\Rightarrow (2z-3)$ is a factor.

Now factorise $f(z)$ using the fact that $(2z-3)$ is a factor.

$$\begin{array}{r l}
 & z^2 - 2z + 5 \\
2z - 3 & \overline{)\,2z^3 - 7z^2 + 16z - 15} \\
 & -(2z^3 - 3z^2) \\
 & -4z^2 + 16z - 15 \\
 & -(-4z^2 + 6z) \\
 & 10z - 15 \\
 & -(10z - 15) \\
 & 0
\end{array}$$

→ Long Division

Solve for z.

So $f(z) = (2z-3)(z^2 - 2z + 5) = 0$

$z = \frac{3}{2}$, $z = \dfrac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times 5}}{2 \times 1}$

$= \dfrac{2 \pm \sqrt{-16}}{2}$

$= 1 \pm 2i$

hence roots of $f(z) = 0$ are

$\frac{3}{2}$, $1 + 2i$, $1 - 2i$

A

4 You are given that $\sum_{r=1}^{n}(ar+b) = n^2$ for all n, where a and b are constants.

By finding $\sum_{r=1}^{n}(ar+b)$ in terms of a, b and n, determine the values of a and b. **[6]**

As stated in the Question, first we find $\sum (ar+b)$

$$\sum_{r=1}^{n} ar+b = a\sum_{r=1}^{n} r + b\sum_{r=1}^{n} 1$$

$$= a\left(\tfrac{1}{2}n(n+1)\right) + b(n)$$

$$= \tfrac{1}{2}an^2 + \tfrac{1}{2}an + bn$$

$$= \tfrac{1}{2}an^2 + \left(\tfrac{1}{2}a+b\right)n$$

So we can write that $\tfrac{1}{2}an^2 + \left(\tfrac{1}{2}a+b\right)n = n^2$

$$\tfrac{1}{2}a = 1 \Rightarrow a = 2$$

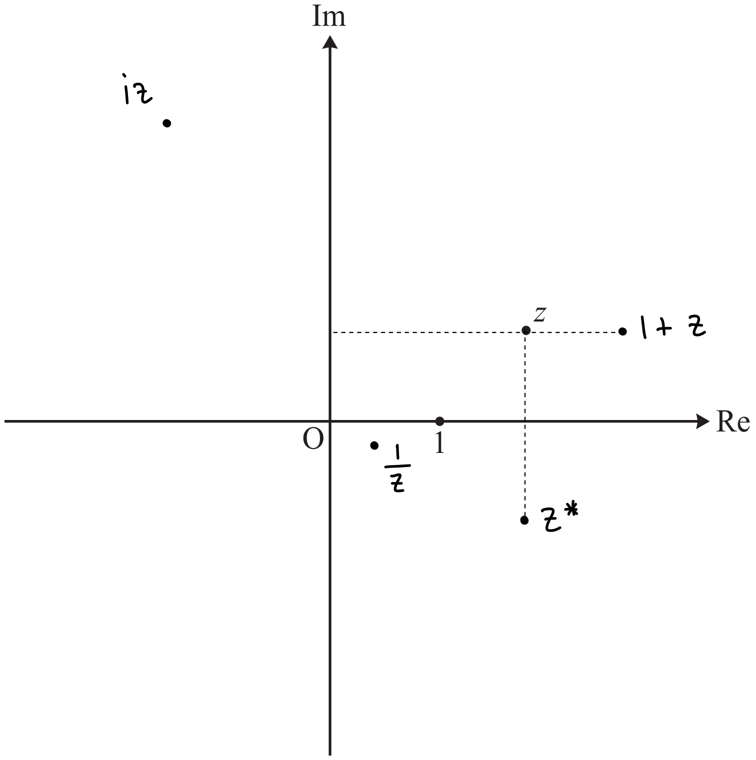
$$\tfrac{1}{2}(2) + b = 0$$

$$b = -1$$

hence $a = 2$, $b = -1$

A

5 The Argand diagram below shows the points representing 1 and z, where $|z| = 2$.



Mark the points representing the following complex numbers on the copy of the diagram in the Printed Answer Booklet, labelling them clearly.

- z^* this is the complex conjugate, so is a reflection of z in the Re axis.
- $\frac{1}{z}$ the new modulus is the reciprocal of the modulus of z.
- $1+z$ this is a horizontal translation of 1 unit to the right.
- iz this is a rotation of z anticlockwise by 90° about the origin.

[4]

G

6 The matrix **M** is $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$.

(a) Calculate $\mathbf{M}^2$, $\mathbf{M}^3$ and $\mathbf{M}^4$. **[2]**

$M^2 = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$

$M^3 = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}$

$M^4 = \begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}$

A

(b) Hence make a conjecture about the matrix $\mathbf{M}^n$. **[1]**

$M^n = \begin{pmatrix} n+1 & n \\ -n & -n+1 \end{pmatrix}$ (using the results in a))

A

(c) Prove your conjecture. ← we can ONLY do this by induction. **[5]**

Step one: base case

When $n=1$, $M^1 = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$, $M^1 = \begin{pmatrix} 1+1 & 1 \\ -1 & -1+1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$

$\therefore$ true for $n=1$

Step two: assumption

Assume true for $n=k$, so $M^k = \begin{pmatrix} k+1 & k \\ -k & -k+1 \end{pmatrix}$

Step three: inductive step

Using the assumed result for $n=k$,

$M^{k+1} = M^k M$

$= \begin{pmatrix} k+1 & k \\ -k & -k+1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 2k+2-k & k+1 \\ -2k+k-1 & -k \end{pmatrix}$

$= \begin{pmatrix} (k+1)+1 & k+1 \\ -(k+1) & -(k+1)+1 \end{pmatrix}$ $\therefore$ true for $n=k+1$

Step four: conclusion

If the result is true for $n=k$, it is true for $n=k+1$. Since it is true for $n=1$, it is true for all positive integer values of n.

G

7 In this question you must show detailed reasoning.

The complex number $\sqrt{3}+\text{i}$ is denoted by z.

(a) By expanding $(\sqrt{3}+\text{i})^5$, express z^5 in the form $a+b\text{i}$ where a and b are real and exact. **[3]**

Expand using a binomial expansion.

$(\sqrt{3}+i)^5 = (\sqrt{3})^5 + {}^5C_1 \times (\sqrt{3})^4 \times (i)^1 + {}^5C_2 \times (\sqrt{3})^3 \times (i)^2$
$+ {}^5C_3 \times (\sqrt{3})^2 \times (i)^3 + {}^5C_4 \times (\sqrt{3}) \times (i)^4$
$+ (i)^5$

$= 9\sqrt{3} + 45i - 30\sqrt{3} - 30i + 5\sqrt{3} + i$

$= -16\sqrt{3} + 16i$

$a = -16\sqrt{3}$

$b = 16$

G

(b) (i) Express z in modulus-argument form. **[3]**

First we find the modulus and argument of z.

$|z| = \sqrt{(\sqrt{3})^2 + (1)^2} = 2$ (using $|z| = \sqrt{a^2+b^2}$)

$\arg z = \arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}$ (using $\arg z = \arctan \frac{b}{a}$)

Now write in $r(\cos\theta + i\sin\theta)$ form

hence $z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

A

(ii) Hence find z^5 in modulus-argument form. **[2]**

Using $\arg z_1 z_2 = \arg z_1 + \arg z_2$,

$\arg z^5 = \arg z + \arg z + \dots + \arg z$

$\arg z^5 = 5 \arg z \Rightarrow \arg z^5 = \frac{5\pi}{6}$

So $z^5 = 2^5\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$z^5 = 32\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

G

(iii) Use this result to verify your answers to part **(a)**. **[2]**

Now we just expand out ii)

$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$ and $\sin\frac{5\pi}{6} = \frac{1}{2}$

hence $z^5 = 32\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -16\sqrt{3} + 16i$ as before

A

8 The equations of three planes are

$$2x + y + 3z = 3,$$
$$3x - y - 2z = 2,$$
$$-4x + 3y + 7z = k,$$

where k is a constant.

(a) By considering a suitable determinant, show that the planes do **not** meet at a single point. **[2]**

In matrix form, $\begin{pmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ -4 & 3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ k \end{pmatrix}$

Using the calculator, the determinant of the coefficient matrix is zero. Hence it is singular, so has no inverse and hence the planes do not meet at a single meet.

R

(b) Given that the planes form a sheaf, determine the value of k. **[4]**

First we need to reduce the system to a system in terms of two unknowns.

plane 1 + plane 2: $2x + y + 3z + 3x - y - 2z = 3 + 2$

$5x + z = 5$ A

3 × plane 2 + plane 3: $9x - 3y - 6z - 4x + 3y + 7z = 6 + k$

$5x + z = 6 + k$ B

For the planes to form a sheaf, A and B must be consistent, so $5 = 6 + k$

$\Rightarrow k = -1$

R

9 A transformation T of the plane is represented by the matrix $\mathbf{M} = \begin{pmatrix} k+1 & -1 \\ 1 & k \end{pmatrix}$, where k is a constant.

Show that, for all values of k, T has no invariant lines through the origin. **[6]**

First we need to set-up $M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$

$$\begin{pmatrix} k+1 & -1 \\ 1 & k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$x(k+1) - y = x'$$

$$x + ky = y'$$

Invariant lines through the origin has the form $y = mx$.
So (x, mx) is mapped to (x', mx').
So substituting in $y' = mx'$ and $y = mx$,

$$x(k+1) - mx = x' \quad \text{(A)}$$

$$x + mkx = mx' \quad \text{(B)}$$

Substituting (A) into (B),

$$x + mkx = m(kx + x - mx)$$

$$x + mkx = mkx + mx - m^2x$$

$$m^2x - mx + x = 0$$

$$x(m^2 - m + 1) = 0$$

So now we need to consider the nature of $m^2 - m + 1 = 0$. If it has no real roots, no invariant lines through the origin exist.

$$\text{discriminant} = (-1)^2 - 4 \times 1 \times 1$$

$$= -3$$

Hence as the discriminant is less than zero, it has no real roots. So M has no invariant lines through the origin. as required

R

10 The plane P has normal vector $2\mathbf{i} + a\mathbf{j} - \mathbf{k}$, where a is a positive constant, and the point $(3, -1, 1)$ lies in P. The plane $x - z = 3$ makes an angle of 45° with P.

Find the cartesian equation of P. **[7]**

First we use the angle between the planes.

Recall that $\cos\theta = \dfrac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|}$.

Since the angle between the planes is 45°, the angle between the normals to the planes is 45°.

Hence

$$\cos 45° = \frac{2(1) + a(0) - 1(-1)}{\sqrt{2^2 + a^2 + 1^2}\sqrt{1^2 + 1^2}}$$

$$\frac{\sqrt{2}}{2} = \frac{3}{\sqrt{5 + a^2}\sqrt{2}}$$

cross multiply

$$2\sqrt{5 + a^2} = 6$$

divide by 2 and square

$$5 + a^2 = 9$$

$$a^2 = 4 \Rightarrow a = \pm 2$$

Since a is positive, $a = 2$.

Recall that the cartesian form of a plane is given by $\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$2x + 2y - z = 6 - 2 - 1$$

$$2x + 2y - z = 3$$ ← this is the required cartesian form of a plane

ADDITIONAL ANSWER SPACE

If additional space is required, you should use the following lined page(s). The question number(s) must be clearly shown in the margin(s).